A note on Fourier-spectral method in JOREK: extending the method for inclusion of second derivatives of the test and basis function

Contents

1 Introduction

In this note the details of the implementation of the Fourier-spectral method in JOREK are written. The aim is to extend the method to include the second derivatives in the physical/numerical model. The physical model can be extended to include the Ohmic heating term. The hyper-diffusion terms in the toroidal direction can be implemented. Furthermore, the complete residual can be taken into account in VMS stabilization terms.

2 Fourier-Spectral method:

Let $\phi = [0, 2\pi[$ denote periodic direction and the governing equation be represented by:

$$\frac{\partial w}{\partial t} = \mathcal{R}(\phi) \qquad\qquad \forall \phi \in [0, 2\pi[$$

The function $\mathcal{R}(\phi)$ is approximated as a truncated Fourier series as:

$$\mathcal{R}(\phi) = \sum_{n=0}^{N} R(n)\ e^{in\phi}$$

where $e^{in\phi}$ are the basis functions. The RHS of a PDE is written in the Galerkin weak form as:

$$I = \int e^{-ik\phi}\ \mathcal{R}(\phi)\ d\phi \qquad\qquad \forall k \in [0, N]$$

Putting the expansion of the function in the above weak form:

$$I = \int e^{-ik\phi}\ \left(\sum_{n=0}^{N} R(n)\ e^{in\phi} \right)\ d\phi \qquad\qquad \forall k \in [0, N]$$

The coefficients $R(n)$ are obtained using discrete Fourier transform. Further,

$$\begin{aligned} I_k &= \int e^{-ik\phi}\ R(n)\ e^{in\phi}\ d\phi \\ &= \int R(n)\ e^{i(n-k)\phi}\ d\phi \\ &= \int R(n)\ e^{i(n-k)\phi}\ d\phi \\ I_k &= 2\pi R(n)\ \delta_{k,n} = 2\pi R(k) \end{aligned}$$

where, $R(k)$ is

$$\begin{aligned} R(k) &= \mathcal{R}(\phi)\ e^{-ik\phi} \\ &= \mathcal{R}(\phi)\cos(k\phi) - i\mathcal{R}(\phi)\sin(k\phi) \\ R(k) &= \mathrm{Real}\big(R(k)\big) - i\ \mathrm{Imag}\big(R(k)\big) \end{aligned}$$

For the first derivatives,

$$\begin{aligned} R(k) &= -i\ k\ \mathcal{R}(\phi)\ e^{-ik\phi} \\ &= -i\ k\ \mathcal{R}(\phi)\cos(\phi) + i^2\ k\ \mathcal{R}(\phi)\sin(\phi) \\ &= -i\ k\ \mathcal{R}(\phi)\cos(\phi) -\ \ k\ \mathcal{R}(\phi)\sin(\phi) \\ R(k) &= -k\ \mathrm{Imag}\big(R(k)\big) - i\ k\ \mathrm{Real}\big(R(k)\big) \end{aligned}$$

For the second derivatives,

$$\begin{aligned}
R(k) &= -k^2 \ \mathcal{R}(\phi) \ e^{-ik\phi} \\
&= -k^2 \ \mathcal{R}(\phi) \cos(\phi) + i \ k^2 \ \mathcal{R}(\phi) \sin(\phi) \\
&= -k^2 \ \mathcal{R}(\phi) \cos(\phi) + i \ k^2 \ \mathcal{R}(\phi) \sin(\phi) \\
R(k) &= -k^2 \ \text{Real}\big(R(k)\big) + i \ k^2 \ \text{Imag}\big(R(k)\big)
\end{aligned}$$

This can be, let us say, RHS_kk is JOREK.

3 Implicit part

Implicit time integration method requires an evaluation of the jacobian matrix. For the RHS:

$$\begin{aligned}
A_{k,m}(\phi) &= \int e^{-ik\phi} \ \frac{\partial \mathcal{R}(\phi)}{\partial w} \ e^{im\phi} \ d\phi \\
&= \int e^{-ik\phi} \ \mathcal{J}(\phi) \ e^{im\phi} \ d\phi \\
&= \int \Big(\cos(k\phi) - i\sin(k\phi) \Big) \ \mathcal{J}(\phi) \ \Big(\cos(m\phi) + i\sin(m\phi) \Big) \ d\phi \\
A_{k,m}(\phi) &= \int \cos(k\phi) \ \mathcal{J}(\phi) \ \cos(m\phi) \ d\phi + \int i \ \cos(k\phi) \ \mathcal{J}(\phi) \ \sin(m\phi) \ d\phi \\
&- \int i\sin(k\phi) \ \mathcal{J}(\phi) \ \cos(m\phi) \ d\phi + \int \ \sin(k\phi) \ \mathcal{J}(\phi) \ \sin(m\phi) \ d\phi
\end{aligned}$$

3.1 No derivatives on the test and basis functions

The first contribution:

$$\begin{aligned}
A^1_{k,m}(\phi) &= \int \cos(k\phi)\ \mathcal{J}(\phi)\ \cos(m\phi)\ d\phi \\
&= \int \cos(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \cos(m\phi)\ d\phi \\
&= \int \frac{1}{2}\Big(e^{ik\phi} + e^{-ik\phi}\Big)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \frac{1}{2}\Big(e^{im\phi} + e^{-im\phi}\Big)\ d\phi \\
&= \frac{1}{4}\ \sum_{l=-N}^{N}\ \int J(l)\Big[e^{i(k+l+m)\phi} + e^{i(k+l-m)\phi} + e^{i(-k+l+m)\phi} + e^{i(-k+l-m)\phi}\Big]\ d\phi \\
&= \frac{1}{4}\ J(l)\ \Big(\delta(k+l+m) + \delta(k+l-m) + \delta(-k+l+m) + \delta(-k+l-m)\Big) \\
&= \frac{1}{4}\ \Big(J(-k-m) + J(-k+m) + J(k-m) + J(k+m)\Big) \\
&= \frac{1}{4}\ \Big(J^*(k+m) + J^*(k-m) + J(k-m) + J(k+m)\Big) \\
A^1_{k,m}(\phi) &= \frac{1}{2}\ \mathrm{Real}\big(J(k+m)\big) + \frac{1}{2}\ \mathrm{Real}\big(J(k-m)\big)
\end{aligned}$$

The second contribution:

$$
\begin{aligned}
A^2_{k,m}(\phi) &= \int \sin(k\phi)\ \mathcal{J}(\phi)\ \cos(m\phi)\ d\phi \\
&= \int \sin(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \cos(m\phi)\ d\phi \\
&= \int \frac{1}{2i}\left(e^{ik\phi} - e^{-ik\phi}\right)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \frac{1}{2}\left(e^{im\phi} + e^{-im\phi}\right)\ d\phi \\
&= \frac{1}{4i}\ \sum_{l=-N}^{N}\ \int J(l)\left[e^{i(k+l+m)\phi} + e^{i(k+l-m)\phi} - e^{i(-k+l+m)\phi} - e^{i(-k+l-m)\phi}\right]\ d\phi \\
&= \frac{1}{4i}\ J(l)\ \Big(\delta(k+l+m) + \delta(k+l-m) - \delta(-k+l+m) - \delta(-k+l-m)\Big) \\
&= \frac{1}{4i}\ \Big(J(-k-m) + J(-k+m) - J(k-m) - J(k+m)\Big) \\
&= \frac{1}{4i}\ \Big(J^*(k+m) + J^*(k-m) - J(k-m) - J(k+m)\Big) \\
&= -\frac{1}{4i}\ \Big(J(k+m) - J^*(k+m)\Big) - \frac{1}{4i}\ \Big(J(k-m) - J^*(k-m)\Big) \\
A^2_{k,m}(\phi) &= -\frac{1}{2}\ \mathrm{Imag}\big(J(k+m)\big) - \frac{1}{2}\ \mathrm{Imag}\big(J(k-m)\big)
\end{aligned}
$$

The third contribution:

$$
\begin{aligned}
A^3_{k,m}(\phi) &= \int \cos(k\phi)\ \mathcal{J}(\phi)\ \sin(m\phi)\ d\phi \\
&= \int \cos(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \sin(m\phi)\ d\phi \\
&= \int \frac{1}{2}\Big(e^{ik\phi} + e^{-ik\phi}\Big)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \frac{1}{2i}\Big(e^{im\phi} - e^{-im\phi}\Big)\ d\phi \\
&= \frac{1}{4i}\ \sum_{l=-N}^{N}\ \int J(l)\Big[e^{i(k+l+m)\phi} - e^{i(k+l-m)\phi} + e^{i(-k+l+m)\phi} - e^{i(-k+l-m)\phi}\Big]\ d\phi \\
&= \frac{1}{4i}\ J(l)\ \Big(\delta(k+l+m) - \delta(k+l-m) + \delta(-k+l+m) - \delta(-k+l-m)\Big) \\
&= \frac{1}{4i}\ \Big(J(-k-m) - J(-k+m) + J(k-m) - J(k+m)\Big) \\
&= \frac{1}{4i}\ \Big(J^*(k+m) - J^*(k-m) + J(k-m) - J(k+m)\Big) \\
&= -\frac{1}{4i}\ \Big(J(k+m) - J^*(k+m)\Big) + \frac{1}{4i}\ \Big(J(k-m) - J^*(k-m)\Big) \\
A^3_{k,m}(\phi) &= -\frac{1}{2}\ \mathrm{Imag}\big(J(k+m)\big) + \frac{1}{2}\ \mathrm{Imag}\big(J(k-m)\big)
\end{aligned}
$$

The fourth contribution:

$$
\begin{aligned}
A^4_{k,m}(\phi) &= \int \sin(k\phi)\ \mathcal{J}(\phi)\ \sin(m\phi)\ d\phi \\
&= \int \sin(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \sin(m\phi)\ d\phi \\
&= \int \frac{1}{2i}\Big(e^{ik\phi} - e^{-ik\phi}\Big)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \frac{1}{2i}\Big(e^{im\phi} - e^{-im\phi}\Big)\ d\phi \\
&= \frac{1}{4i^2}\ \sum_{l=-N}^{N}\ \int J(l)\Big[e^{i(k+l+m)\phi} - e^{i(k+l-m)\phi} - e^{i(-k+l+m)\phi} + e^{i(-k+l-m)\phi}\Big]\ d\phi \\
&= -\frac{1}{4}\ J(l)\ \Big(\delta(k+l+m) - \delta(k+l-m) - \delta(-k+l+m) + \delta(-k+l-m)\Big) \\
&= -\frac{1}{4}\ \Big(J(-k-m) - J(-k+m) - J(k-m) + J(k+m)\Big) \\
&= -\frac{1}{4}\ \Big(J^*(k+m) - J^*(k-m) - J(k-m) + J(k+m)\Big) \\
&= -\frac{1}{4}\ \Big(J(k+m) + J^*(k+m)\Big) + \frac{1}{4}\ \Big(J(k-m) + J^*(k-m)\Big) \\
A^4_{k,m}(\phi) &= -\frac{1}{2}\ \mathrm{Real}\big(J(k+m)\big) + \frac{1}{2}\ \mathrm{Real}\big(J(k-m)\big)
\end{aligned}
$$

$A^1_{k,m}(\phi)$, $A^2_{k,m}(\phi)$, $A^3_{k,m}(\phi)$ and $A^4_{k,m}(\phi)$ derived above are the contribution to ELM in JOREK.

3.2 First derivative on the basis function

The first contribution:

$$
\begin{aligned}
A^1_{k,m}(\phi) &= \int \cos(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial}{\partial\phi}\cos(m\phi)\ d\phi \\
&= -m\int \cos(k\phi)\ \mathcal{J}(\phi)\ \sin(m\phi)\ d\phi \\
&= -m\Big[-\frac{1}{2}\ \mathrm{Imag}\big(J(k+m)\big) + \frac{1}{2}\ \mathrm{Imag}\big(J(k-m)\big)\Big] \\
A^1_{k,m}(\phi) &= \frac{m}{2}\ \mathrm{Imag}\big(J(k+m)\big) - \frac{m}{2}\ \mathrm{Imag}\big(J(k-m)\big)
\end{aligned}
$$

The second contribution:

$$
\begin{aligned}
A^2_{k,m}(\phi) &= \int \sin(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial}{\partial\phi}\cos(m\phi)\ d\phi \\
&= -m \int \sin(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \ \sin(m\phi)\ d\phi \\
&= -m\Big[-\frac{1}{2}\ \text{Real}\big(J(k+m)\big) + \frac{1}{2}\ \text{Real}\big(J(k-m)\big)\Big] \\
A^2_{k,m}(\phi) &= \frac{m}{2}\ \text{Real}\big(J(k+m)\big) - \frac{m}{2}\ \text{Real}\big(J(k-m)\big)\Big]
\end{aligned}
$$

The third contribution:

$$
\begin{aligned}
A^3_{k,m}(\phi) &= \int \cos(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial}{\partial\phi}\sin(m\phi)\ d\phi \\
&= m \int \cos(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \ \cos(m\phi)\ d\phi \\
&= m\Big[\frac{1}{2}\ \text{Real}\big(J(k+m)\big) + \frac{1}{2}\ \text{Real}\big(J(k-m)\big)\Big] \\
A^3_{k,m}(\phi) &= \frac{m}{2}\ \text{Real}\big(J(k+m)\big) + \frac{m}{2}\ \text{Real}\big(J(k-m)\big)
\end{aligned}
$$

The fourth contribution:

$$
\begin{aligned}
A^4_{k,m}(\phi) &= \int \sin(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial}{\partial\phi}\sin(m\phi)\ d\phi \\
&= m \int \sin(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \ \cos(m\phi)\ d\phi \\
&= m\Big[-\frac{1}{2}\ \text{Imag}\big(J(k+m)\big) - \frac{1}{2}\ \text{Imag}\big(J(k-m)\big)\Big] \\
A^4_{k,m}(\phi) &= -\frac{m}{2}\ \text{Imag}\big(J(k+m)\big) - \frac{m}{2}\ \text{Imag}\big(J(k-m)\big)
\end{aligned}
$$

$A^1_{k,m}(\phi)$, $A^2_{k,m}(\phi)$, $A^3_{k,m}(\phi)$ and $A^4_{k,m}(\phi)$ derived above are the contribution to ELM_n in JOREK.

3.3 First derivative on the test function

The first contribution:

$$
\begin{aligned}
A^1_{k,m}(\phi) &= \int \frac{\partial}{\partial\phi}\cos(k\phi)\ \mathcal{J}(\phi)\ \cos(m\phi)\ d\phi \\
&= -m\int \sin(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \cos(m\phi)\ d\phi \\
&= -m\Big[-\frac{1}{2}\ \mathrm{Imag}\big(J(k+m)\big) - \frac{1}{2}\ \mathrm{Imag}\big(J(k-m)\big)\Big] \\
A^1_{k,m}(\phi) &= \frac{m}{2}\ \mathrm{Imag}\big(J(k+m)\big) + \frac{m}{2}\ \mathrm{Imag}\big(J(k-m)\big)
\end{aligned}
$$

The second contribution:

$$
\begin{aligned}
A^2_{k,m}(\phi) &= \int \frac{\partial}{\partial\phi}\sin(k\phi)\ \mathcal{J}(\phi)\ \cos(m\phi)\ d\phi \\
&= m\int \cos(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \cos(m\phi)\ d\phi \\
&= m\Big[\frac{1}{2}\ \mathrm{Real}\big(J(k+m)\big) + \frac{1}{2}\ \mathrm{Real}\big(J(k-m)\big)\Big] \\
A^2_{k,m}(\phi) &= \frac{m}{2}\ \mathrm{Real}\big(J(k+m)\big) + \frac{m}{2}\ \mathrm{Real}\big(J(k-m)\big)
\end{aligned}
$$

The third contribution:

$$
\begin{aligned}
A^3_{k,m}(\phi) &= \int \frac{\partial}{\partial\phi}\cos(k\phi)\ \mathcal{J}(\phi)\ \sin(m\phi)\ d\phi \\
&= -m\int \sin(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \sin(m\phi)\ d\phi \\
&= -m\Big[-\frac{1}{2}\ \mathrm{Real}\big(J(k+m)\big) + \frac{1}{2}\ \mathrm{Real}\big(J(k-m)\big)\Big] \\
A^3_{k,m}(\phi) &= \frac{m}{2}\ \mathrm{Real}\big(J(k+m)\big) - \frac{m}{2}\ \mathrm{Real}\big(J(k-m)\big)
\end{aligned}
$$

The fourth contribution:

$$
\begin{aligned}
A^4_{k,m}(\phi) &= \int \frac{\partial}{\partial\phi}\sin(k\phi)\ \mathcal{J}(\phi)\ \sin(m\phi)\ d\phi \\
&= m\int \cos(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \sin(m\phi)\ d\phi \\
&= m\Big[-\frac{1}{2}\ \mathrm{Imag}\big(J(k+m)\big) + \frac{1}{2}\ \mathrm{Imag}\big(J(k-m)\big)\Big] \\
A^4_{k,m}(\phi) &= -\frac{m}{2}\ \mathrm{Imag}\big(J(k+m)\big) + \frac{m}{2}\ \mathrm{Imag}\big(J(k-m)\big)
\end{aligned}
$$

$A^1_{k,m}(\phi)$, $A^2_{k,m}(\phi)$, $A^3_{k,m}(\phi)$ and $A^4_{k,m}(\phi)$ derived above are the contribution to ELM_k in JOREK.

3.4 First derivative on the basis and the test function

The first contribution:

$$\begin{aligned}
A^1_{k,m}(\phi) &= \int \frac{\partial}{\partial\phi}\cos(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial}{\partial\phi}\cos(m\phi)\ d\phi \\
&= k\ m \int \sin(k\phi)\ \mathcal{J}(\phi)\ \sin(m\phi)\ d\phi \\
&= k\ m\Big[-\frac{1}{2}\ \mathrm{Real}\big(J(k+m)\big) + \frac{1}{2}\ \mathrm{Real}\big(J(k-m)\big)\Big] \\
A^1_{k,m}(\phi) &= -\frac{k\ m}{2}\ \mathrm{Real}\big(J(k+m)\big) + \frac{k\ m}{2}\ \mathrm{Real}\big(J(k-m)\big)
\end{aligned}$$

The second contribution:

$$\begin{aligned}
A^2_{k,m}(\phi) &= \int \frac{\partial}{\partial\phi}\sin(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial}{\partial\phi}\cos(m\phi)\ d\phi \\
&= -km \int \cos(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \sin(m\phi)\ d\phi \\
&= -km\Big[-\frac{1}{2}\ \mathrm{Imag}\big(J(k+m)\big) + \frac{1}{2}\ \mathrm{Imag}\big(J(k-m)\big)\Big] \\
A^2_{k,m}(\phi) &= \frac{km}{2}\ \mathrm{Imag}\big(J(k+m)\big) - \frac{km}{2}\ \mathrm{Imag}\big(J(k-m)\big)
\end{aligned}$$

The third contribution:

$$\begin{aligned}
A^3_{k,m}(\phi) &= \int \frac{\partial}{\partial\phi}\cos(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial}{\partial\phi}\sin(m\phi)\ d\phi \\
&= -km \int \sin(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \cos(m\phi)\ d\phi \\
&= -km\Big[-\frac{1}{2}\ \mathrm{Imag}\big(J(k+m)\big) - \frac{1}{2}\ \mathrm{Imag}\big(J(k-m)\big)\Big] \\
A^3_{k,m}(\phi) &= \frac{km}{2}\ \mathrm{Imag}\big(J(k+m)\big) + \frac{km}{2}\ \mathrm{Imag}\big(J(k-m)\big)\Big]
\end{aligned}$$

The fourth contribution:

$$\begin{aligned}
A^4_{k,m}(\phi) &= \int \frac{\partial}{\partial\phi}\sin(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial}{\partial\phi}\sin(m\phi)\ d\phi \\
&= km\int \cos(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \cos(m\phi)\ d\phi \\
&= km\Big[\frac{1}{2}\ \mathrm{Real}\big(J(k+m)\big) + \frac{1}{2}\ \mathrm{Real}\big(J(k-m)\big)\Big] \\
A^4_{k,m}(\phi) &= \frac{km}{2}\ \mathrm{Real}\big(J(k+m)\big) + \frac{km}{2}\ \mathrm{Real}\big(J(k-m)\big)
\end{aligned}$$

$A^1_{k,m}(\phi)$, $A^2_{k,m}(\phi)$, $A^3_{k,m}(\phi)$ and $A^4_{k,m}(\phi)$ derived above are the contribution to ELM_n_k in JOREK.

3.5 Second derivative on the basis function

The first contribution:

$$\begin{aligned}
A^1_{k,m}(\phi) &= \int \cos(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial^2}{\partial\phi^2}\cos(m\phi)\ d\phi \\
&= -m^2\int \cos(k\phi)\ \mathcal{J}(\phi)\ \cos(m\phi)\ d\phi \\
&= -m^2\Big[\frac{1}{2}\ \mathrm{Real}\big(J(k+m)\big) + \frac{1}{2}\ \mathrm{Real}\big(J(k-m)\big)\Big] \\
A^1_{k,m}(\phi) &= -\frac{m^2}{2}\ \mathrm{Real}\big(J(k+m)\big) - \frac{m^2}{2}\ \mathrm{Real}\big(J(k-m)\big)
\end{aligned}$$

The second contribution:

$$\begin{aligned}
A^2_{k,m}(\phi) &= \int \sin(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial^2}{\partial\phi^2}\cos(m\phi)\ d\phi \\
&= -m^2\int \sin(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \cos(m\phi)\ d\phi \\
&= -m^2\Big[-\frac{1}{2}\ \mathrm{Imag}\big(J(k+m)\big) - \frac{1}{2}\ \mathrm{Imag}\big(J(k-m)\big)\Big] \\
A^2_{k,m}(\phi) &= \frac{m^2}{2}\ \mathrm{Imag}\big(J(k+m)\big) + \frac{m^2}{2}\ \mathrm{Imag}\big(J(k-m)\big)
\end{aligned}$$

The third contribution:

$$\begin{aligned}
A^3_{k,m}(\phi) &= \int \cos(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial^2}{\partial\phi^2}\sin(m\phi)\ d\phi \\
&= -m^2 \int \cos(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \sin(m\phi)\ d\phi \\
&= -m^2 \Big[-\frac{1}{2}\ \mathrm{Imag}\big(J(k+m)\big) + \frac{1}{2}\ \mathrm{Imag}\big(J(k-m)\big)\Big] \\
A^3_{k,m}(\phi) &= \frac{m^2}{2}\ \mathrm{Imag}\big(J(k+m)\big) - \frac{m^2}{2}\ \mathrm{Imag}\big(J(k-m)\big)
\end{aligned}$$

The fourth contribution:

$$\begin{aligned}
A^4_{k,m}(\phi) &= \int \sin(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial^2}{\partial\phi^2}\sin(m\phi)\ d\phi \\
&= -m^2 \int \sin(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \sin(m\phi)\ d\phi \\
&= -m^2 \Big[-\frac{1}{2}\ \mathrm{Real}\big(J(k+m)\big) + \frac{1}{2}\ \mathrm{Real}\big(J(k-m)\big)\Big] \\
A^4_{k,m}(\phi) &= \frac{m^2}{2}\ \mathrm{Real}\big(J(k+m)\big) - \frac{m^2}{2}\ \mathrm{Real}\big(J(k-m)\big)
\end{aligned}$$

$A^1_{k,m}(\phi)$, $A^2_{k,m}(\phi)$, $A^3_{k,m}(\phi)$ and $A^4_{k,m}(\phi)$ derived above can be the contribution to ELM_nn in JOREK.

3.6 Second derivative on the test function

The first contribution:

$$\begin{aligned}
A^1_{k,m}(\phi) &= \int \frac{\partial^2}{\partial\phi^2}\cos(k\phi)\ \mathcal{J}(\phi)\cos(m\phi)\ d\phi \\
&= -k^2 \int \cos(k\phi)\ \mathcal{J}(\phi)\ \cos(m\phi)\ d\phi \\
&= -k^2 \Big[\frac{1}{2}\ \mathrm{Real}\big(J(k+m)\big) + \frac{1}{2}\ \mathrm{Real}\big(J(k-m)\big)\Big] \\
A^1_{k,m}(\phi) &= -\frac{k^2}{2}\ \mathrm{Real}\big(J(k+m)\big) - \frac{k^2}{2}\ \mathrm{Real}\big(J(k-m)\big)
\end{aligned}$$

The second contribution:

$$\begin{aligned}
A^2_{k,m}(\phi) &= \int \frac{\partial^2}{\partial\phi^2}\sin(k\phi)\ \mathcal{J}(\phi)\cos(m\phi)\ d\phi \\
&= -k^2\int \sin(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \cos(m\phi)\ d\phi \\
&= -k^2\Big[-\frac{1}{2}\ \mathrm{Imag}\big(J(k+m)\big) - \frac{1}{2}\ \mathrm{Imag}\big(J(k-m)\big)\Big] \\
A^2_{k,m}(\phi) &= \frac{k^2}{2}\ \mathrm{Imag}\big(J(k+m)\big) + \frac{k^2}{2}\ \mathrm{Imag}\big(J(k-m)\big)
\end{aligned}$$

The third contribution:

$$\begin{aligned}
A^3_{k,m}(\phi) &= \int \frac{\partial^2}{\partial\phi^2}\cos(k\phi)\ \mathcal{J}(\phi)\sin(m\phi)\ d\phi \\
&= -k^2\int \cos(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \sin(m\phi)\ d\phi \\
&= -k^2\Big[-\frac{1}{2}\ \mathrm{Imag}\big(J(k+m)\big) + \frac{1}{2}\ \mathrm{Imag}\big(J(k-m)\big)\Big] \\
A^3_{k,m}(\phi) &= \frac{k^2}{2}\ \mathrm{Imag}\big(J(k+m)\big) - \frac{k^2}{2}\ \mathrm{Imag}\big(J(k-m)\big)
\end{aligned}$$

The fourth contribution:

$$\begin{aligned}
A^4_{k,m}(\phi) &= \int \frac{\partial^2}{\partial\phi^2}\sin(k\phi)\ \mathcal{J}(\phi)\sin(m\phi)\ d\phi \\
&= -k^2\int \sin(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \sin(m\phi)\ d\phi \\
&= -k^2\Big[-\frac{1}{2}\ \mathrm{Real}\big(J(k+m)\big) + \frac{1}{2}\ \mathrm{Real}\big(J(k-m)\big)\Big] \\
A^4_{k,m}(\phi) &= \frac{k^2}{2}\ \mathrm{Real}\big(J(k+m)\big) - \frac{k^2}{2}\ \mathrm{Real}\big(J(k-m)\big)
\end{aligned}$$

$A^1_{k,m}(\phi)$, $A^2_{k,m}(\phi)$, $A^3_{k,m}(\phi)$ and $A^4_{k,m}(\phi)$ derived above can be the contribution to ELM_kk in JOREK.

3.7 First derivative on the test function and second derivative on the basis function

The first contribution:

$$\begin{aligned} A^1_{k,m}(\phi) &= \int \frac{\partial}{\partial \phi} \cos(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial^2}{\partial \phi^2} \cos(m\phi)\ d\phi \\ &= km^2 \int \sin(k\phi)\ \mathcal{J}(\phi)\ \cos(m\phi)\ d\phi \\ &= km^2 \Big[-\frac{1}{2}\ \text{Imag}\big(J(k+m)\big) - \frac{1}{2}\ \text{Imag}\big(J(k-m)\big) \Big] \\ A^1_{k,m}(\phi) &= -\frac{k\ m^2}{2}\ \text{Imag}\big(J(k+m)\big) - \frac{k\ m^2}{2}\ \text{Imag}\big(J(k-m)\big) \end{aligned}$$

The second contribution:

$$\begin{aligned} A^2_{k,m}(\phi) &= \int \frac{\partial}{\partial \phi} \sin(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial^2}{\partial \phi^2} \cos(m\phi)\ d\phi \\ &= -km^2 \int \cos(k\phi)\ \sum_{l=-N}^{N} J(l) e^{il\phi}\ \cos(m\phi)\ d\phi \\ &= -km^2 \Big[\frac{1}{2}\ \text{Real}\big(J(k+m)\big) + \frac{1}{2}\ \text{Real}\big(J(k-m)\big) \Big] \\ A^2_{k,m}(\phi) &= -\frac{km^2}{2}\ \text{Real}\big(J(k+m)\big) - \frac{km^2}{2}\ \text{Real}\big(J(k-m)\big) \end{aligned}$$

The third contribution:

$$\begin{aligned} A^3_{k,m}(\phi) &= \int \frac{\partial}{\partial \phi} \cos(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial^2}{\partial \phi^2} \sin(m\phi)\ d\phi \\ &= k\ m^2 \int \sin(k\phi)\ \sum_{l=-N}^{N} J(l) e^{il\phi}\ \sin(m\phi)\ d\phi \\ &= k\ m^2 \Big[-\frac{1}{2}\ \text{Real}\big(J(k+m)\big) + \frac{1}{2}\ \text{Real}\big(J(k-m)\big) \Big] \\ A^3_{k,m}(\phi) &= -\frac{k\ m^2}{2}\ \text{Real}\big(J(k+m)\big) + \frac{k\ m^2}{2}\ \text{Real}\big(J(k-m)\big) \end{aligned}$$

The fourth contribution:

$$\begin{aligned} A^4_{k,m}(\phi) &= \int \frac{\partial}{\partial \phi} \sin(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial^2}{\partial \phi^2} \sin(m\phi)\ d\phi \\ &= -k\ m^2 \int \cos(k\phi)\ \sum_{l=-N}^{N} J(l) e^{il\phi}\ \sin(m\phi)\ d\phi \\ &= -k\ m^2 \Big[-\frac{1}{2}\ \text{Imag}\big(J(k+m)\big) + \frac{1}{2}\ \text{Imag}\big(J(k-m)\big) \Big] \\ A^4_{k,m}(\phi) &= \frac{k\ m^2}{2}\ \text{Imag}\big(J(k+m)\big) - \frac{k\ m^2}{2}\ \text{Imag}\big(J(k-m)\big) \end{aligned}$$

$A^1_{k,m}(\phi)$, $A^2_{k,m}(\phi)$, $A^3_{k,m}(\phi)$ and $A^4_{k,m}(\phi)$ derived above can be the contribution to ELM_n_kk in JOREK.

3.8 Second derivative on the test function and first derivative on the basis function

an be the contribution to ELM_nn_k in JOREK.

3.9 Second derivative on the test function and second derivative on the basis function

The first contribution:

$$\begin{aligned}
A^1_{k,m}(\phi) &= \int \frac{\partial^2}{\partial\phi^2}\cos(k\phi)\ \mathcal{J}(\phi)\ \ \frac{\partial^2}{\partial\phi^2}\cos(m\phi)\ d\phi \\
&= k^2m^2\int \cos(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \cos(m\phi)\ d\phi \\
&= k^2m^2\Big[\frac{1}{2}\ \mathrm{Real}\big(J(k+m)\big)+\frac{1}{2}\ \mathrm{Real}\big(J(k-m)\big)\Big] \\
A^1_{k,m}(\phi) &= \frac{k^2m^2}{2}\ \mathrm{Real}\big(J(k+m)\big)+\frac{k^2m^2}{2}\ \mathrm{Real}\big(J(k-m)\big)
\end{aligned}$$

The second contribution:

$$\begin{aligned}
A^2_{k,m}(\phi) &= \int \frac{\partial^2}{\partial\phi^2}\sin(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial^2}{\partial\phi^2}\cos(m\phi)\ d\phi \\
&= k^2m^2\int \sin(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \cos(m\phi)\ d\phi \\
&= k^2m^2\Big[-\frac{1}{2}\ \mathrm{Imag}\big(J(k+m)\big)-\frac{1}{2}\ \mathrm{Imag}\big(J(k-m)\big)\Big] \\
A^2_{k,m}(\phi) &= -\frac{k^2m^2}{2}\ \mathrm{Imag}\big(J(k+m)\big)-\frac{k^2m^2}{2}\ \mathrm{Imag}\big(J(k-m)\big)
\end{aligned}$$

The third contribution:

$$\begin{aligned}
A^3_{k,m}(\phi) &= \int \frac{\partial^2}{\partial\phi^2}\cos(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial^2}{\partial\phi^2}\sin(m\phi)\ d\phi \\
&= k^2m^2\int \cos(k\phi)\ \sum_{l=-N}^{N} J(l)e^{il\phi}\ \sin(m\phi)\ d\phi \\
&= k^2m^2\Big[-\frac{1}{2}\ \mathrm{Imag}\big(J(k+m)\big)+\frac{1}{2}\ \mathrm{Imag}\big(J(k-m)\big)\Big] \\
A^3_{k,m}(\phi) &= -\frac{k^2m^2}{2}\ \mathrm{Imag}\big(J(k+m)\big)+\frac{k^2m^2}{2}\ \mathrm{Imag}\big(J(k-m)\big)
\end{aligned}$$

The fourth contribution:

$$\begin{aligned}
A^4_{k,m}(\phi) &= \int \frac{\partial^2}{\partial \phi^2}\sin(k\phi)\ \mathcal{J}(\phi)\ \frac{\partial^2}{\partial \phi^2}\sin(m\phi)\ d\phi \\
&= k^2 m^2 \int \sin(k\phi)\ \sum_{l=-N}^{N} J(l) e^{il\phi}\ \sin(m\phi)\ d\phi \\
&= k^2 m^2 \Big[-\frac{1}{2}\ \mathrm{Real}\big(J(k+m)\big) + \frac{1}{2}\ \mathrm{Real}\big(J(k-m)\big)\Big] \\
A^4_{k,m}(\phi) &= -\frac{k^2 m^2}{2}\ \mathrm{Real}\big(J(k+m)\big) + \frac{k^2 m^2}{2}\ \mathrm{Real}\big(J(k-m)\big)
\end{aligned}$$

$A^1_{k,m}(\phi)$, $A^2_{k,m}(\phi)$, $A^3_{k,m}(\phi)$ and $A^4_{k,m}(\phi)$ derived above are the contribution to ELM_nn_kk in JOREK.

4 Ohmic Heating

The ohmic heating term in weak form is:

$$RHS = V_i\ \eta\ \boldsymbol{J} \cdot \boldsymbol{J}$$

The implicit part of the term:

$$\begin{aligned}
\frac{\partial RHS}{\partial A_i} &= V_i\ \eta\ 2\ \boldsymbol{J} \cdot \frac{\partial \boldsymbol{J}}{\partial A_i} \qquad\qquad i = 1,2,3 \\
\frac{\partial RHS}{\partial T} &= V_i\ \frac{\partial \eta}{\partial T}\ 2\ \boldsymbol{J} \cdot \boldsymbol{J}
\end{aligned}$$

Components of the current vector,

$$\begin{aligned}
J_R =& \frac{\partial B_\phi}{\partial Z} - \frac{1}{R}\frac{\partial B_Z}{\partial \phi} \\
J_Z =& \frac{1}{R}\left(\frac{\partial B_R}{\partial \phi} - \frac{\partial (R B_\phi)}{\partial R}\right) \\
J_\phi =& \left(\frac{\partial B_Z}{\partial R} - \frac{\partial B_R}{\partial Z}\right)
\end{aligned}$$

Components of the current in terms of the components of the magnetic potential:

$$J_R = \left(\frac{\partial^2 A_Z}{\partial Z \partial R} - \frac{\partial^2 A_R}{\partial Z^2} + \frac{1}{R}\frac{\partial F}{\partial \psi}\frac{\partial \psi}{\partial Z}\right) - \frac{1}{R^2}\left(\frac{\partial^2 A_R}{\partial \phi^2} - \frac{\partial^2 \psi}{\partial \phi \partial R}\right)$$

$$
\begin{aligned}
J_Z =& \frac{1}{R^2}\left(\frac{\partial^2\psi}{\partial\phi\partial Z} - \frac{\partial^2 A_Z}{\partial\phi^2}\right) - \left(\frac{\partial^2 A_Z}{\partial R^2} - \frac{\partial^2 A_R}{\partial R\partial Z} + \frac{1}{R}\frac{\partial F}{\partial\psi}\frac{\partial\psi}{\partial R} - \frac{F}{R^2}\right)\\
&- \frac{1}{R}\left(\frac{\partial A_Z}{\partial R} - \frac{\partial A_R}{\partial Z} + \frac{F(\psi)}{R}\right)
\end{aligned}
$$

$$
J_\phi = \frac{1}{R}\left(\frac{\partial^2 A_R}{\partial R\partial\phi} - \frac{\partial^2\psi}{\partial R^2}\right) - \frac{1}{R^2}\left(\frac{\partial A_R}{\partial\phi} - \frac{\partial\psi}{\partial R}\right) - \frac{1}{R}\left(\frac{\partial^2\psi}{\partial Z^2} - \frac{\partial^2 A_Z}{\partial Z\partial\phi}\right)
$$

In the implicit part, there will be terms containing second derivatives of the basis function and hence ELM_nn needs to be implemented.

New contributions required: ELM_nn.