# Shock capturing stabilization terms in JOREK

Galerkin Finite element method (FEM) gives centered approximations of differential operators. When the flow is convection dominated, a centered approximation gives rise to spurious waves which can adversely affect the stability and accuracy of the numerical methods. In this context, the numerical stabilization is needed. The stabilization of a Galerkin FEM is achieved, in a generalized framework using Variable Multiscale (VMS) formulation [5]. VMS framework accommodates SUPG [7, 3], Taylor-Galerkin stabilization [2] approaches as special cases. For the reduced MHD models used in JOREK, Taylor-Galerkin stabilization terms have already been implemented [4].

VMS based stabilization are $\mathcal{L}^2$, meaning that, they act when a numerical solution is smooth. In presence of shocks/discontinuities in a numerical solution, Gibbs phenomenon can give rise to spurious oscillations. These oscillations can also adversely affect the stability and accuracy of the numerical methods. VMS based stabilization do not provide the mechanism to stabilize such spurious oscillations. Therefore, an addition stabilization mechanism for bounded variation of a numerical solution is needed which acts locally, only near the shocks/discontinuities. Historically, such stabilization is called as 'shock-capturing stabilization'. Focus of this note is on the Shock-capturing stabilization.

The shock/discontinuity detection criterion forms the crucial part of the shock capturing stabilization strategy. The notable strategy developed in [6] use the pressure gradient as a shock-detection criterion and the numerical stabilization is added near shocks to stabilize the centered finite volume approximation. In the literature wide-range of choices are available to detect different types of shock/discontinuities based on different variables such as pressure or entropy [7, 1, 3]. Shock-detection based on the pressure gradient is one of the widely used choice to design the shock-capturing methods. Further, the stabilization terms can be added in some preferred direction. Since JOREK deals with strongly magnetized plasma simulation, the direction parallel to the magnetic field is targeted for the shock-capturing strategy.

# 1 Physical model

MHD equations can be written compactly below as the system of partial differential equations.

$$\mathbb{M}\ \partial_t \boldsymbol{w}\ +\ \mathbb{A}\nabla\boldsymbol{w}\ =\ \nabla\cdot\boldsymbol{d}\ +\ \dot{\boldsymbol{s}}$$

where, $\boldsymbol{w}$ is a vector of variables, $\mathbb{M}$ denotes the mass matrix, $\mathbb{A}\nabla\boldsymbol{w}$ denote convective terms, $\nabla\cdot\boldsymbol{d}$ denote diffusive terms and $\dot{\boldsymbol{s}}$ denote sources. If the right hand side is zero, one recovers the ideal MHD model.

The vector of variables for the reduced MHD model600 in JOREK is as follows:

$$\boldsymbol{w} = \{w_i\}^T = \{\psi,\ u,\ j_\phi,\ w,\ \rho,\ v_\parallel,\ T_i,\ T_e,\ \rho_n\}^T$$

and the same for the full MHD model712 is as follows:

$$\boldsymbol{w} = \{w_i\}^T = \{\boldsymbol{A},\ \boldsymbol{v},\ \rho,\ \ T_i,\ T_e,\ \rho_n\}^T$$

# 2 Weak form

In the framework of Galerkin FEM, the weak form the equations can be written as:

$$\mathbb{M}\ \partial_t \boldsymbol{w}\ +\ \mathbb{A}\nabla\boldsymbol{w}\ =\ \nabla\cdot\boldsymbol{d}\ +\ \dot{\boldsymbol{s}}$$

The matrix $\mathbb{M}$ is the mass matrix, the vector $\mathbb{A}\nabla\boldsymbol{w}$ denote convective terms, the vector $\nabla\cdot\boldsymbol{d}$ denote diffusive terms and the vector $\dot{\boldsymbol{s}}$ denote source terms.

The weak form can be further written compactly by introducing the definition of the residual $\mathcal{R}(\boldsymbol{w}) = \mathbb{M}\ \partial_t\boldsymbol{w} + \mathbb{A}\nabla\boldsymbol{w} - \nabla\cdot\boldsymbol{d} - \dot{\boldsymbol{s}}$ as:

$$\Big(\boldsymbol{w}^*,\ \mathcal{R}(\boldsymbol{w})\Big) = 0 \tag{1}$$

where $(.,.)$ denotes $\mathcal{L}^2$ product and $\boldsymbol{w}^*$ are the test functions. Equation (1) denotes the FEM applied to the physical models in JOREK. The numerical stabilization is realized by additional terms in the weak form as:

$$\Big(\boldsymbol{w}^*,\ \mathcal{R}(\boldsymbol{w})\Big) + \boldsymbol{D}_{\text{vms}} + \boldsymbol{D}_{\text{sc}} = 0 \tag{2}$$

where, $\boldsymbol{D}_{\text{vms}}$ denotes the VMS based stabilization and $\boldsymbol{D}_{\text{sc}}$ denotes the shock-capturing stabilization.

# 3 VMS/Taylor-Galerkin stabilization

Talylor-Galerkin stabilization can be thought as a special case of VMS based stabilization and is written as:

$$\boldsymbol{D}_{\text{vms}} = \sum_{e=1}^{N_e} \int \left(\mathbb{A}\nabla \boldsymbol{w}^*\right) \cdot \left(\mathbb{A}\nabla \boldsymbol{w}\right) \, d\Omega \tag{3}$$

This is the stabilization implemented in reduced MHD models in JOREK. The index $e$ runs over all the elements.

# 4 Shock capturing terms

Shock capturing stabilization scheme to be applied for reduced MHD model, in the framework of Galerkin FEM, is written in the weak form as follows:

$$\boldsymbol{D}_{\text{sc}} = -\sum_{e=1}^{Ne} \int \nabla \boldsymbol{w}^* : \boldsymbol{d}_{num} \, d\Omega_e \tag{4}$$

Fluxes $\boldsymbol{d}_{num}$ can be formulated to make the stabilization parallel to the magnetic field. Therefore, the stabilization terms in each equation become:

$$D_i = -\sum_{e=1}^{Ne} \int \nabla w^* : \left(\tau_{\parallel,i} \; \nabla_{\parallel} w_i + \tau_{\perp,i} \; \nabla_{\perp} w_i\right) d\Omega_e \qquad \forall i \in 1, ..., N_v \tag{5}$$

where, $N_v$ is the number of variables. Using the definition of the parallel gradient:

$$D_i = -\sum_{e=1}^{Ne} \int \left[ (\tau_{\parallel,i} - \tau_{\perp,i}) \; (\boldsymbol{b} \cdot \nabla w^*)(\boldsymbol{b} \cdot \nabla w_i) \; + \; \tau_{\perp,i} \; \nabla w^* \cdot \nabla w_i \right] d\Omega_e \tag{6}$$

**Remarks:**

1) The form of shock-capturing terms is similar to physical diffusivities and hence the shock-capturing terms can be thought as additional diffusion.

2) The aim is to add diffusion/numerical stabilization only near shocks and hence shock-detection forms the crucial part of the scheme.

3) The coefficient of the shock-capturing terms is designed such that the stabilization will act locally, only in the region of shocks. Moreover, the stabilization should go to zero with grid refinement.

# 5 Application to reduced MHD

The choice of the numerical coefficients $\tau_{\|,i}$ and $\tau_{\perp,i}$ is as follows:

$$\tau_{\|,i} = C_{\|,i}\ h^2\ \frac{|d_p| + |d_s|}{p}\ f_p \tag{7}$$

$$\tau_{\perp,i} = C_{\perp,i}\ h^2\ \frac{|d_p| + |d_s|}{p}\ f_p \tag{8}$$

$h$ being the element size. $C_{\|,i}$ and $C_{\perp,i}$ are real numbers and the user defined parameters to be set via input file. The design of the shock-detection criteria is motivated by that used in [1, 3].

The non-dimensional term $f_p$ is given by:

$$f_p = \frac{||\nabla p||}{p}\ h \tag{9}$$

where $p$ is the pressure. This term is an example of shock-detector. At the location of the hydrodynamic shocks, $f_p$ takes high values locally. The element size $h$ makes the term $f_p$ non-dimensional.

All other terms in the equations (8), apart from $f_p$, provides the strength the stabilization. The term $d_p$ is given by:

$$d_p = \sum_{i=1}^{N_v} \frac{\partial p}{\partial w_i}\ \tilde{\mathcal{R}}(w_i)$$

where, $\tilde{\mathcal{R}}(w_i)$ are the approximated residual in the equation for the variable $w_i$. By design, this term is based on the pressure related variables only. The term $d_s$ is given by:

$$d_s = \sum_{i=1}^{N_v} \frac{\partial p}{\partial w_i}\ \dot{w}_i \tag{10}$$

where, $\dot{w}_i$ is a source term in the equation of $w_i$. The dimensions of $\tau_{\|,i}$ and $\tau_{\perp,i}$ then mimic those of physical diffusivities.

Remarks specific to implementation in reduced MHD:

1) With neutrals, the total pressure is defined as

$$p = p_i + p_e + \rho_n\ T_i \tag{11}$$

with the assumption that the neutrals are at the ion temperature $T_i$.

2) In practice, the term $d_p$ is implemented as follows:

$$d_p = T\ \tilde{\mathcal{R}}(\rho)\ +\ \tilde{\mathcal{R}}(p)\ +\ T\ \tilde{\mathcal{R}}(\rho_n) \tag{12}$$

For the two temperature model,

$$d_p = (T_i + T_e)\ \tilde{\mathcal{R}}(\rho)\ +\ \tilde{\mathcal{R}}(p_i)\ +\ \tilde{\mathcal{R}}(p_e)\ +\ (T_i + T_e)\ \tilde{\mathcal{R}}(\rho_n) \tag{13}$$

The residuals are approximated as:

$$\begin{aligned}
\tilde{\mathcal{R}}(\rho) &\approx \nabla \cdot (\rho\ \boldsymbol{v}) \\
\tilde{\mathcal{R}}(\rho_n) &\approx \nabla \cdot (\rho_n\ \boldsymbol{v}) \\
\tilde{\mathcal{R}}(p) &\approx \boldsymbol{v} \cdot \nabla p + \gamma\ p\ \nabla \cdot \boldsymbol{v}
\end{aligned}$$

3) Addition of the term $d_s$ is kept optional and can be seen as taking 'more complete' residual into account. This term may be activated in case of strong sources.

   In the branch, the logical variable `add_sources_in_sc` can be used to activate the terms, which is set to false by default.

3) The terms $C_{\|,i}$ and $C_{\perp,i}$ are user defined parameters to control the amount of the stabilization and set to zero by default. To add the stabilization strictly in the parallel (to the magnetic field) direction, only $C_{\|,i}$ need to be used. To add 'isotropic' shock-capturing stabilization, $C_{\|,i} = C_{\perp,i}$ should be set.

   The interpretation and the implementation of these coefficients is as follows:

$$\begin{aligned}
&\mu \rightarrow \mu + \mu_{\perp,sc} \\
&\mu_{\|} \rightarrow \mu_{\|} + \mu_{\|,sc} \\
&D_{\perp} \rightarrow D_{\perp} + D_{\perp,sc} \\
&D_{\|} \rightarrow D_{\|} + D_{\|,sc} \\
&\kappa_{\perp} \rightarrow \kappa_{\perp} + \kappa_{\perp,sc} \\
&\kappa_{\|} \rightarrow \kappa_{\|} + \kappa_{\|,sc}
\end{aligned}$$

   In the branch, their names given are `D_par_sc_num`, `D_perp_sc_num` and so on.

4) Numerical values for $C_{\|,i}$ and $C_{\perp,i}$ are grid and the physical problem dependent and hence their tuning will be required. As a starting point, $C_{\|,i} = 100$ and $C_{\perp,i} = 0$ may be used.

5) Stabilization terms are not added in the equation of $\psi$, as the design of the shock-detection criterion is based on hydrodynamic pressure.

6) Stabilization terms are not added in the equations of $j_\phi$ and vorticity as these equations do not have convection terms.

# 6 Demonstration: model600

# References


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