Hierarchy of fluids models for magnetized and collisional plasmas.

Part II: Reduced MHD model

E. Franck,* J. Morales,† A. Lessig ‡,
C. Caldini-Queiros ‡, M. Hölzl ‡

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Contents

1 Full MHD proposed for JOREK

Firstly we propose to recall the full model considered. This model contains a large part of the physic present in JOREK. Adding the source terms and the neoclassical tensors we obtain the physical behavior of JOREK.

[1]Inria Nancy grand Est, TONUS Team, Strasbourg, France
[2]CEA Cadarache, IRFM, France
[3]Max-Planck-Institut für Plasmaphysik, Boltzmannstraße 2D-85748 Garching, Germany.

Bi-pressure MHD for tokamak with simplify tensors

$$
\begin{cases}
\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\
\rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{J} \times \boldsymbol{B} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{gv} \\
\partial_t \frac{1}{\gamma - 1} p_i + \frac{1}{\gamma - 1} \boldsymbol{u} \cdot \nabla p_i + \frac{\gamma}{\gamma - 1} p_i \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q}_i + \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} : \nabla \boldsymbol{u} \\
= 3 \frac{\rho_e}{\tau_e m_i} (T_i - T_e) \\
\partial_t \frac{1}{\gamma - 1} p_e + \frac{1}{\gamma - 1} \boldsymbol{u} \cdot \nabla p_e + \frac{\gamma}{\gamma - 1} p_e \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q}_e \\
= \frac{1}{\gamma - 1} \frac{m_i}{\rho e} \boldsymbol{J} \cdot \left(\nabla p_e - \gamma p_e \frac{\nabla \rho}{\rho} \right) - 3 \frac{\rho_e}{\tau_e m_i} (T_i - T_e) + \eta |\boldsymbol{J}|^2 \\
\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E} \\
\boldsymbol{E} = \left(-\boldsymbol{u} \times \boldsymbol{B} + \eta \boldsymbol{J} - \frac{m_i}{\rho e} \nabla p_e + \frac{m_i}{\rho e} (\boldsymbol{J} \times \boldsymbol{B}) \right) \\
\mu_0 \nabla \times \boldsymbol{B} = \boldsymbol{J}, \quad \nabla \cdot \boldsymbol{B} = 0
\end{cases}
\tag{1.1}
$$

with

$$
\begin{cases}
\boldsymbol{u} = \boldsymbol{u}_{\perp} + \boldsymbol{u}_{\parallel}, \quad \boldsymbol{u}_{\perp} = \boldsymbol{u}_E + \boldsymbol{u}_i^*, \quad \boldsymbol{u}_i^* = \frac{m_i}{e \rho} \frac{\boldsymbol{B} \times \nabla p_i}{|\, \boldsymbol{B} \,|^2}, \quad \boldsymbol{u}_E = \frac{\boldsymbol{E} \times \boldsymbol{B}}{|\, \boldsymbol{B} \,|^2}, \quad \boldsymbol{u}_{\parallel} = v_{\parallel} \frac{\boldsymbol{B}}{|\, \boldsymbol{B} \,|} \\
\nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{gv} = -\rho \boldsymbol{u}_i^* \cdot \nabla \boldsymbol{u} + p_i \left(\nabla \times \frac{m_i \mathbf{b}}{e \parallel \boldsymbol{B} \parallel} \right) \cdot \nabla \boldsymbol{u} \\
+ \nabla_{\perp} \left(\frac{m_i p_i}{2e \parallel \boldsymbol{B} \parallel} \nabla \cdot \mathbf{b} \times \boldsymbol{u} \right) + \mathbf{b} \times \nabla \left(\frac{m_i p_i}{2e \parallel \boldsymbol{B} \parallel} \nabla_{\perp} \cdot \boldsymbol{u} \right) \\
\nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} = G \mathbf{b} \cdot \nabla \mathbf{b} - \frac{1}{3} \nabla G + \nabla_{\parallel} G, \quad \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} : \nabla \boldsymbol{u} = -\frac{1}{3 \eta_0} G^2 \\
G = -\eta_0 (2 \mathbf{b} \cdot \nabla v_{\parallel} - ((\mathbf{b} \cdot \nabla \mathbf{b}) \cdot \boldsymbol{u}_{\perp}))
\end{cases}
\tag{1.2}
$$

After have recall this model we proposed to write the reduced version. This version will gives a alternative to the JOREK model 333 which the conservation or dissipation to the energy. Consequently we propose to write the reduction for each equation and after check the energy balance. For the reduction we assume that

Reduction theory

- $\boldsymbol{B} = \frac{F_0}{R}\boldsymbol{e}_\phi + \frac{1}{R}\nabla\psi \times \boldsymbol{e}_\phi$
- $\boldsymbol{E} = -\nabla F_0 u$

We define also the operator used in the computation

Differential operator

- $\nabla = \partial_R \boldsymbol{e}_R + \partial_Z \boldsymbol{e}_Z + \frac{1}{R}\partial_\phi \boldsymbol{e}_\phi$
- $\nabla_{pol} = \partial_R \boldsymbol{e}_R + \partial_Z \boldsymbol{e}_Z$
- $\nabla\cdot = \frac{1}{R}\partial_R(R)\boldsymbol{e}_R + \partial_Z \boldsymbol{e}_Z + \frac{1}{R^2}\partial_\phi \boldsymbol{e}_\phi$
- $\Delta_{pol} = \frac{1}{R}\partial_R(R\partial_R)\boldsymbol{e}_R + \partial_{ZZ}\boldsymbol{e}_Z$
- $\Delta^* = R\partial_R(\frac{1}{R}\partial_R)\boldsymbol{e}_R + \partial_{ZZ}\boldsymbol{e}_Z$

These operator are used in all the part of the reduced MHD. Now to simplify the computation off the reduced model we introduce some definition of the classical quantities

Definition of fields

- $\mid \boldsymbol{B} \mid^2 = \dfrac{1}{R^2}\left(F_0^2 + \mid \nabla\psi \mid^2\right) \approx \dfrac{F_0^2}{R^2}$
- $\boldsymbol{u}_E = R\partial_R u \boldsymbol{e}_Z - R\partial_Z u \boldsymbol{e}_R + \left(\dfrac{1}{F_0}\left(R(\nabla_{pol}u, \nabla_{pol}\psi)\boldsymbol{e}_\phi - \partial_\phi u \nabla_{pol}\psi\right)\right)$
- $\boldsymbol{u}_i^* = \dfrac{m_i}{e\rho}\left(R\partial_R p_i \boldsymbol{e}_Z - R\partial_Z p_i \boldsymbol{e}_R\right) + \left(\dfrac{m_i}{F_0 e\rho}\left(R(\nabla_{pol}p_i, \nabla_{pol}\psi)\boldsymbol{e}_\phi - \partial_\phi p_i \nabla_{pol}\psi\right)\right)$
- $\boldsymbol{J} = j_\phi \boldsymbol{e}_\phi + \boldsymbol{J}_{pol} = -\left(\dfrac{1}{R}\Delta^*\psi\boldsymbol{e}_\phi - \dfrac{1}{R^2}\nabla_{pol}(\partial_\phi\psi)\right)$

Proof. The two first are trivial. We details the computation of the $\boldsymbol{u}_E = \frac{\boldsymbol{E}\times\boldsymbol{B}}{|\boldsymbol{B}|^2}$

$$
\begin{aligned}
\boldsymbol{E}\times\boldsymbol{B} &= F_0\left(-\partial_R u \boldsymbol{e}_R - \partial_Z u \boldsymbol{e}_Z - \frac{1}{R}\partial_\phi u \boldsymbol{e}_\phi\right) \times \left(\frac{F_0}{R}\boldsymbol{e}_\phi + (\frac{1}{R}\partial_Z\psi\boldsymbol{e}_R - \frac{1}{R}\partial_R\psi\boldsymbol{e}_Z)\right) \\
&= \frac{F_0^2}{R}\left(\partial_R u \boldsymbol{e}_Z - \partial_Z u \boldsymbol{e}_R\right) - \frac{F_0}{R^2}\left(\partial_R\Psi\partial_\phi u \boldsymbol{e}_R + \partial_Z\Psi\partial_\phi u \boldsymbol{e}_Z\right) + \frac{F_0}{R}(\nabla u, \nabla\psi)\boldsymbol{e}_\phi
\end{aligned}
$$

We finish using the approximation of norm to the magnetic field $\mid \boldsymbol{B} \mid^2 \approx \frac{F_0^2}{R^2}$. To obtain the approximation of the diamagnetic velocity we use that

$$
\boldsymbol{B}\times\nabla p_i = -\nabla p_i \times \boldsymbol{B}
$$

and we use the previous formula. For the current it is details in the article of the single fluid reduced MHD. ■

We introduce also classical operator which appears in the computation

Reduction Advection operator

- $\boldsymbol{B}\cdot\nabla f = \dfrac{F_0}{R^2}\partial_\phi f - \dfrac{1}{R}[\psi, f]$ and $\boldsymbol{u}_{\parallel}\cdot\nabla f = \dfrac{\boldsymbol{B}}{\mid\boldsymbol{B}\mid}\cdot\nabla f$
- $\boldsymbol{u}_E\cdot\nabla f = R[u,f] + \left(\dfrac{1}{F_0}\left((\nabla_{pol}u, \nabla_{pol}\psi)\partial_\phi f - \partial_\phi u(\nabla_{pol}\psi, \nabla_{pol}f)\right)\right)$
- $\boldsymbol{u}_i^*\cdot\nabla f = \dfrac{F_0\tau}{R\rho}[p_i, f] + \left(\dfrac{\tau}{R\rho}\left((\nabla_{pol}p_i, \nabla_{pol}\psi)\partial_\phi f - \partial_\phi p_i(\nabla_{pol}\psi, \nabla_{pol}f)\right)\right)$

with $\tau = \frac{m_i}{F_0 e}$

Reduction divergence operator

- $\nabla\cdot(\boldsymbol{u}_{\parallel}) = \boldsymbol{B}\cdot\nabla v_{\parallel}$
- $\nabla\cdot\boldsymbol{u}_E = -2\rho\partial_Z u + \dfrac{1}{F_0}\left[\partial_\phi(\nabla_{pol}u, \nabla_{pol}\psi) - (\nabla_{pol}u, \nabla_{pol}\partial_\phi\psi) - \Delta_{pol}\psi\partial_\phi u\right]$
- $\nabla\cdot\boldsymbol{u}_i^* = -2\rho\partial_Z p_i + \dfrac{1}{F_0}\left[\partial_\phi(\nabla_{pol}p_i, \nabla_{pol}\psi) - (\nabla_{pol}p_i, \nabla_{pol}\partial_\phi\psi) - \Delta_{pol}\psi\partial_\phi p_i\right]$

2 Induction equation

We propose in this section to write the reduced induction equation. for this we consider the full one before.

Full induction equation

$$\partial_t\boldsymbol{B} = -\nabla\times\left(-\boldsymbol{u}\times\boldsymbol{B} + \eta\boldsymbol{J} - \frac{m_i}{\rho e}\nabla p_e + \frac{m_i}{\rho e}(\boldsymbol{J}\times\boldsymbol{B})\right)$$

The induction equation is the following

$$\frac{\partial\boldsymbol{B}}{\partial t} = -\nabla\times\boldsymbol{E}. \tag{2.3}$$

Using the vector potential $\boldsymbol{A}$ the induction equation writes

$$\frac{\partial\boldsymbol{A}}{\partial t} = -\boldsymbol{E} - \nabla\Phi, \tag{2.4}$$

with Φ the scalar electric potential. Knowing that in the reduced MHD approach only the toroidal component of the vector potential varies we can write Eq. (2.4) as a function of the magnetic flux ψ as follows

$$\partial_t\left(\frac{\psi}{R}\right)\boldsymbol{e}_\phi = -\boldsymbol{E} - F_0\nabla u. \tag{2.5}$$

We write the electric potential $\Phi = F_0\, u$, in this way we simplify the final formulation of the $\boldsymbol{E}\times\boldsymbol{B}$ velocity ($\boldsymbol{u}_E$). The expression for the electric field $\boldsymbol{E}$ can be derived from the generalized Ohm's law write in the model (1.1)

$$\partial_t\left(\frac{\psi}{R}\right)\boldsymbol{e}_\phi = \frac{m_i}{e\rho}\nabla p_e + \boldsymbol{u}\times\boldsymbol{B} - \frac{m_i}{e\rho}(\boldsymbol{J}\times\boldsymbol{B}) - \eta\boldsymbol{J} - F_0\nabla u \tag{2.6}$$

To obtain the equation on ψ we project in the $\boldsymbol{B}$ direction the previous equation. We obtain

$$F_0\partial_t\left(\frac{\psi}{R^2}\right) = \tau\frac{F_0}{\rho}\boldsymbol{B}\cdot\nabla p_e - \eta\boldsymbol{B}\cdot\boldsymbol{J} + F_0\boldsymbol{B}\cdot\nabla u$$

The computation of $\boldsymbol{B}\cdot\boldsymbol{J}$ gives

At the end we have

Reduced induction equation

$$\partial_t\left(\frac{\psi}{R^2}\right) = \frac{\tau}{\rho}\boldsymbol{B}\cdot\nabla p_e + \frac{\eta}{R^2}\Delta^*\psi + \boldsymbol{B}\cdot\nabla u + \mathcal{R}_{\psi,j}$$

with new small term $\mathcal{R}_{\psi,j} = -\frac{\eta}{R^3}\left([\partial_\phi\psi,\psi]\right)$

3 Toroidal current equation and perpendicular vorticity equation

4 Density equation

Now we propose to compute the density equation

Full density equation

$$\partial_t\rho + \nabla\cdot(\rho\boldsymbol{u}) = 0$$

Firstly we decompose in two part

$$\partial_t\rho + \rho\nabla\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\rho = 0$$

Now we separate the velocity field to obtain the final results

Reduced density equation

$$\partial_t\rho + \rho\left(\boldsymbol{B}\cdot\nabla v_\parallel + \nabla\cdot\boldsymbol{u}_E + \nabla\cdot\boldsymbol{u}_i^*\right) + \left(v_\parallel\boldsymbol{B} + \boldsymbol{u}_E + \boldsymbol{u}_i^*\right)\cdot\nabla\rho = 0$$

5 Ion pressure equation

Now we propose to compute the density equation

Full ion pressure equation

$$\partial_t \frac{1}{\gamma-1} p_i + \frac{1}{\gamma-1} \boldsymbol{u} \cdot \nabla p_i + \frac{\gamma}{\gamma-1} p_i \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q}_i + \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} : \nabla \boldsymbol{u} = 3 \frac{\rho_e}{\tau_e m_i} (T_i - T_e)$$

Firstly we decompose in two part

$$\partial_t \rho + \rho \nabla \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \rho = 0$$

As for the density Equation we have

$$\boldsymbol{u} \cdot \nabla p_i = \left(v_{\parallel} \boldsymbol{B} + \boldsymbol{u}_E + \boldsymbol{u}_i^* \right) \cdot \nabla p_i = 0$$

and

$$p_i \nabla \cdot \boldsymbol{u} = \left(\boldsymbol{B} \cdot \nabla v_{\parallel} + \nabla \cdot \boldsymbol{u}_E + \nabla \cdot \boldsymbol{u}_i^* \right) p_i$$

The heating flux is given by

$$\mathbf{q}_i = \nabla \cdot \left((k_{\parallel} - k_{\perp}) \frac{\boldsymbol{B}}{\mid \boldsymbol{B} \mid^2} (\boldsymbol{B} \cdot \nabla T_i) + k_{\perp} \nabla T_i \right)$$

Now the last term is the viscous parallel heating which depend of G. To begin we try to compute the field line curvature term $\mathbf{b} \cdot \nabla \mathbf{b}$. Firstly we assume that $\mid \boldsymbol{B} \mid = \frac{F0}{R}$ consequently $\nabla(\frac{1}{|\boldsymbol{B}|}) \approx \frac{1}{F_0}$. We have

$$\mathbf{b} \cdot \nabla \mathbf{b} = \frac{R}{F_0^2} \boldsymbol{B} \cdot \nabla \boldsymbol{B} \approx \frac{R}{F_0^2} \boldsymbol{B} \cdot \nabla \left(\frac{F_0}{R^2} \right) \boldsymbol{e}_{\phi} + \frac{R}{F_0^2} \boldsymbol{B} \cdot \nabla \left(\frac{1}{R} \nabla \psi \times \boldsymbol{e}_{\phi} \right)$$

$$\begin{aligned}
\overline{\overline{\boldsymbol{\Pi}}}_{\parallel} : \nabla \boldsymbol{u} &= -\frac{1}{3\eta_0} G^2 \\
&= \frac{\eta_0}{3} (2\mathbf{b} \cdot \nabla v_{\parallel} - ((\mathbf{b} \cdot \nabla \mathbf{b}) \cdot \boldsymbol{u}_{\perp}))^2 \\
&= \frac{\eta_0}{3} (2\mathbf{b} \cdot \nabla v_{\parallel} - \frac{R}{F_0^2} (\boldsymbol{B} \cdot \nabla \boldsymbol{B}) \cdot \boldsymbol{u}_{\perp}))^2
\end{aligned}$$

We obtain this first writing of the pressure equation

Reduced ion pressure equation

$$\begin{aligned}
&\partial_t \frac{1}{\gamma-1} p_i + \frac{1}{\gamma-1} \left(v_{\parallel} \boldsymbol{B} + \boldsymbol{u}_E + \boldsymbol{u}_i^* \right) \cdot \nabla p_i + \frac{\gamma}{\gamma-1} \left(\boldsymbol{B} \cdot \nabla v_{\parallel} + \nabla \cdot \boldsymbol{u}_E + \nabla \cdot \boldsymbol{u}_i^* \right) p_i \\
&+ \nabla \cdot \left((k_{\parallel} - k_{\perp}) \frac{\boldsymbol{B}}{\mid \boldsymbol{B} \mid^2} (\boldsymbol{B} \cdot \nabla T_i) + k_{\perp} \nabla T_i \right) + \frac{\eta_0}{3} (2\mathbf{b} \cdot \nabla v_{\parallel} - \frac{R}{F_0^2} (\boldsymbol{B} \cdot \nabla \boldsymbol{B}) \cdot \boldsymbol{u}_{\perp}))^2 \\
&= 3 \frac{\rho_e}{\tau_e m_i} (T_i - T_e)
\end{aligned}$$

6 Electron pressure equation

Now we propose to compute the density equation

Full electron pressure equation

$$\partial_t \frac{1}{\gamma-1} p_e + \frac{1}{\gamma-1} \boldsymbol{u} \cdot \nabla p_e + \frac{\gamma}{\gamma-1} p_e \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q}_e$$
$$= \frac{1}{\gamma-1} \frac{m_i}{\rho e} \boldsymbol{J} \cdot \left(\nabla p_e - \gamma p_e \frac{\nabla \rho}{\rho} \right) - 3 \frac{\rho_e}{\tau_e m_i} (T_i - T_e) + \eta |\boldsymbol{J}|^2$$

Firstly we decompose in two part

$$\partial_t \rho + \rho \nabla \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \rho = 0$$

As for the density Equation we have

$$\boldsymbol{u} \cdot \nabla p_e = \left(v_\parallel \boldsymbol{B} + \boldsymbol{u}_E + \boldsymbol{u}_i^* \right) \cdot \nabla p_e = 0$$

and

$$p_e \nabla \cdot \boldsymbol{u} = \left(\boldsymbol{B} \cdot \nabla v_\parallel + \nabla \cdot \boldsymbol{u}_E + \nabla \cdot \boldsymbol{u}_i^* \right) p_e$$

The heating flux is given by

$$\mathbf{q}_e = \nabla \cdot \left((k_\parallel - k_\perp) \frac{\boldsymbol{B}}{|\boldsymbol{B}|^2} (\boldsymbol{B} \cdot \nabla T_e) + k_\perp \nabla T_e \right)$$

For the Ohmic heating and the term which depend of the electronic pressure, we separate the part linked to the toroidal and poloidal current. Ad the end we will try to keep on only the part associated to the toroidal current. We obtain

Reduced electronic pressure equation

$$\partial_t \frac{1}{\gamma-1} p_e + \frac{1}{\gamma-1} \left(v_\parallel \boldsymbol{B} + \boldsymbol{u}_E + \boldsymbol{u}_i^* \right) \cdot \nabla p_e + \frac{\gamma}{\gamma-1} \left(\boldsymbol{B} \cdot \nabla v_\parallel + \nabla \cdot \boldsymbol{u}_E + \nabla \cdot \boldsymbol{u}_i^* \right) p_e$$
$$+ \nabla \cdot \left((k_\parallel - k_\perp) \frac{\boldsymbol{B}}{|\boldsymbol{B}|^2} (\boldsymbol{B} \cdot \nabla T_e) + k_\perp \nabla T_e \right) - \frac{\eta}{R^2} |\Delta^* \psi|^2$$
$$= \frac{1}{\gamma-1} \frac{\tau}{R\rho} \Delta^* \psi \left(\partial_\phi p_e - \gamma p_e \frac{\partial_\phi \rho}{\rho} \right) - 3 \frac{\rho_e}{\tau_e m_i} (T_i - T_e) + \mathcal{R}_{p,j}$$

with new small term

$$\mathcal{R}_{p,j} = \frac{\eta}{R^4} \mid \nabla_{pol} (\partial_\phi \psi) \mid^2 + \frac{1}{\gamma-1} \frac{\tau}{R^2 \rho} \nabla_{pol} (\partial_\phi \psi) \cdot \left(\nabla_{pol} p_e - \gamma p_e \frac{\nabla_{pol} \rho}{\rho} \right)$$

7 Parallel momentum equation

For the parallel momentum equation we multiply the momentum equation by $\mathbf{b}$.

8 Perpendicular momentum equation